Homogeneization: Affine to Linear setup

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1 Homogeneization of variables

It is frequent to face a affine expression of variables of the form

$$w_0 + \sum_{i=1}^{D} w_i \boldsymbol{x}_{[i]} = 0 \qquad\qquad \boldsymbol{x} \in \mathbb{R}^D \tag{1}$$

A major example is linear classification where $\{w_i\}_{i=1}^{D}$ represent the parameters of a separating hyperplane and w_0 is the bias. The same appears in linear regression or optimisation of the weights of a artificial neuron.

Equation (1) can be rewritten in vector form as $w_0 + \boldsymbol{w}^\mathsf{T}\boldsymbol{x} = 0$ ($\boldsymbol{w}, \boldsymbol{x} \in \mathbb{R}^D$) but the bias is not well integrated and this form is still not easy to manipulate. Instead one can define homogeneous variables $\check{\boldsymbol{x}} \in \mathbb{R}^{D+1}$ and $\check{\boldsymbol{w}} \in \mathbb{R}^{D+1}$ so that equation (1) is equivalent to:

$$\begin{cases} \check{\boldsymbol{w}}^\mathsf{T}\check{\boldsymbol{x}} = 0 \\ \check{\boldsymbol{x}}_{[0]} = 1 \\ \check{\boldsymbol{x}}_{[i]} = \boldsymbol{x}_{[i]} \quad i \in [\![1, D]\!] \end{cases} \tag{2}$$

Definition 1 (Homogeneous coordinates)**.** Given variable $\boldsymbol{x} \in \mathbb{R}^D$, the homoegeneous variable $\check{\boldsymbol{x}} \in \mathbb{R}^{D+1}$ has homogeneous coordinates

$$\begin{cases} \check{\boldsymbol{x}}_{[0]} = 1 \\ \check{\boldsymbol{x}}_{[i]} = \boldsymbol{x}_{[i]} \qquad i \in [\![1, D]\!] \end{cases}$$

Geometrically, the homogeneization of variables immerses the Euclidean space $\mathbb{R}^D$ into the space[a] $\mathbb{R}^{D+1}$ and characterizes the points satisfying equation (1) as a D-dimensional hyperplane being the interstection of two $(D+1)$-dimensional hyperplanes described by equation (2)

Example 1 (Illustration $D = 2$)**.** Using classical affine algebra in the Euclidean plane we have

$$w_2\boldsymbol{x}_{[2]} + w_1\boldsymbol{x}_{[1]} + w_0 = 0 \quad \Rightarrow \quad \boldsymbol{x}_{[2]} = -\frac{w_1}{w_2}\boldsymbol{x}_{[1]} - \frac{w_0}{w_2}$$

so that the line $(\mathbb{L}) \subset \mathbb{R}^D$ in the Euclidean plane is

$$(\mathbb{L}) \ : \ \boldsymbol{x}_{[2]} = a\boldsymbol{x}_{[1]} + b \quad \text{where} \quad a = -\frac{w_1}{w_2} \ \text{ and } \ b = -\frac{w_0}{w_2}$$

In the homegeneous space (projective plane) we obtain two planes[b] $\mathbb{P}, \mathbb{M} \subset \mathbb{R}^{D+1}$

$$\begin{cases} (\mathbb{P}) \ : \ \check{\boldsymbol{w}}_{[2]}\check{\boldsymbol{x}}_{[2]} + \check{\boldsymbol{w}}_{[1]}\check{\boldsymbol{x}}_{[1]} + \check{\boldsymbol{w}}_{[0]}\check{\boldsymbol{x}}_{[0]} = \check{\boldsymbol{w}}^\mathsf{T}\check{\boldsymbol{x}} = 0 \\ (\mathbb{M}) \ : \ \check{\boldsymbol{x}}_{[0]} = 1 \end{cases} \quad \text{so that} \quad \mathbb{L} = \mathbb{P} \cap \mathbb{M}$$

Note that

$$\boldsymbol{x} \in \mathbb{L} \quad \Longleftrightarrow \quad \check{\boldsymbol{x}} \in \mathbb{P} \quad \Longleftrightarrow \quad \check{\boldsymbol{w}}^\mathsf{T}\check{\boldsymbol{x}} = 0$$

so that vector $\check{\boldsymbol{w}}$ is a normal to the hyperplane $\mathbb{P}$.

The homogeneous representation is therefore very useful in Machine Learning since we are interested in considering $\mathbb{L}$ as a separating hyperplane. A data $\boldsymbol{x}_i$ is therefore in one side or the other of this hyperplane depending on the angle $\angle(\check{\boldsymbol{x}}_i, \check{\boldsymbol{w}})$ between its homogeneized version $\check{\boldsymbol{x}}_i$ and the normal $\check{\boldsymbol{w}}$ to plane $\mathbb{P}$

$$\begin{cases} \angle(\check{\boldsymbol{x}}_i, \check{\boldsymbol{w}}) < \frac{\pi}{2} & \Leftrightarrow \quad \check{\boldsymbol{x}}_i^\mathsf{T}\check{\boldsymbol{w}} < 0 \quad \rightarrow \quad \text{class } 0 \\ \angle(\check{\boldsymbol{x}}_i, \check{\boldsymbol{w}}) > \frac{\pi}{2} & \Leftrightarrow \quad \check{\boldsymbol{x}}_i^\mathsf{T}\check{\boldsymbol{w}} > 0 \quad \rightarrow \quad \text{class } 1 \end{cases} \tag{3}$$

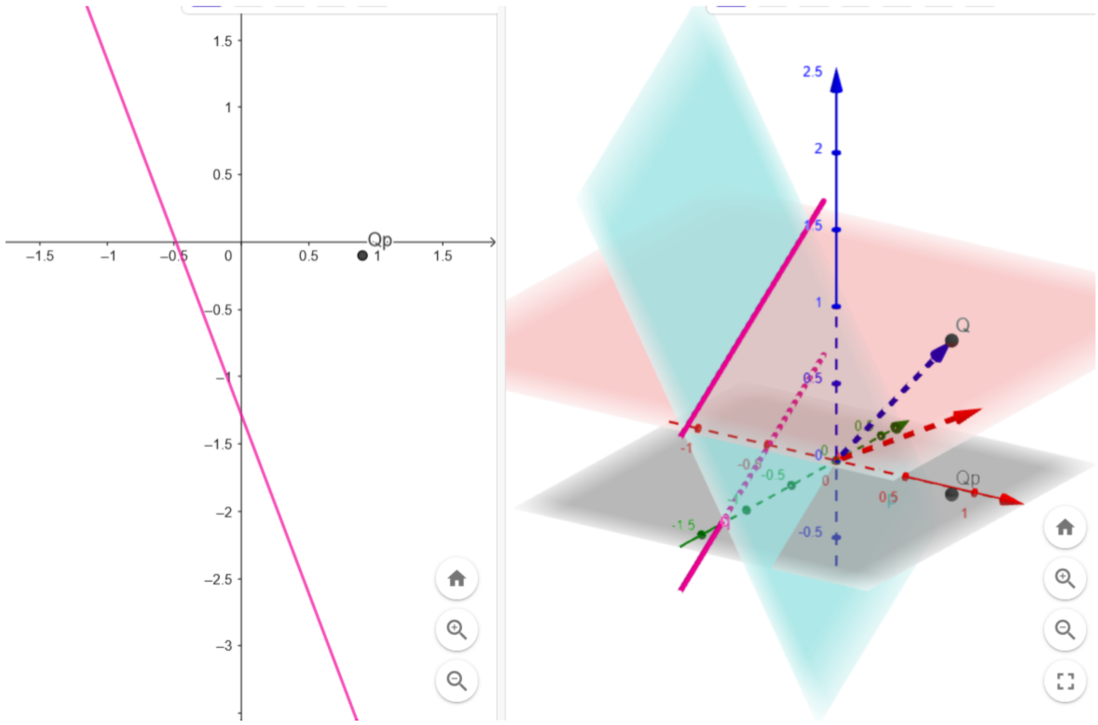

Figure 1: Homogeneization of variables ($D = 2$)

Note. The classification test in equation (3) is <u>not</u> equivalent to the test in non-homogeneous coordinates:

$$\begin{cases} \angle(\mathbf{x}_i, \mathbf{w}) < \frac{\pi}{2} & \Leftrightarrow \quad \mathbf{x}_i^T \mathbf{w} < 0 \quad \not\Rightarrow \quad \text{class } 0 \\ \angle(\mathbf{x}_i, \mathbf{w}) > \frac{\pi}{2} & \Leftrightarrow \quad \mathbf{x}_i^T \mathbf{w} > 0 \quad \not\Rightarrow \quad \text{class } 1 \end{cases} \tag{4}$$

The above test rather tests whether the data is on one side or the other of a line parallel to the pink line in figure 1 (left - 2D setup) but going thru the (2D) origin.

Example 2 (Illustration $D = 1$)**.** The above can also be illustrated in the case where $D = 1$, which corresponds to thresholding $\mathbf{x}_{[1]}$. Let $a \in \mathbb{R}$ be the value of the threshold.

The homogeneous equivalent of equation $\mathbf{x}_{[1]} = a$ reads

$$\mathring{\mathbf{w}}^T \mathring{\mathbf{x}} = \mathring{\mathbf{w}}_{[1]} \mathring{\mathbf{x}}_{[1]} + \mathring{\mathbf{w}}_{[0]} \mathring{\mathbf{x}}_{[0]} = 0 \quad \Rightarrow \quad (\mathbb{P}) \; : \; \mathring{\mathbf{x}}_{[0]} = -\frac{\mathring{\mathbf{w}}_{[1]}}{\mathring{\mathbf{w}}_{[0]}} \mathring{\mathbf{x}}_{[1]} \qquad (\mathbb{M}) \; : \; \mathring{\mathbf{x}}_{[0]} = 1$$

and

$$\mathbb{L} = \mathbb{P} \cap \mathbb{M} \quad \Rightarrow \quad (\mathbb{L}) \; : \; \mathbf{x}_{[1]} = a \quad \Rightarrow \quad a = -\frac{\mathring{\mathbf{w}}_{[0]}}{\mathring{\mathbf{w}}_{[1]}}$$

Figure 2: Homogeneization of variables ($D = 1$) - left as exercise until filled by the author

[a] called <u>projective plane</u> if $D = 2$